Definition

Set $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$ in $\mathbb{R}^n$.

The vectors $\mathbf{e}_1, \ldots, \mathbf{e}_n$ are called the standard basis for $\mathbb{R}^n$.

Note that $\text{span}\{\mathbf{e}_1, \ldots, \mathbf{e}_n\} = \mathbb{R}^n$, since

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = a_1 \cdot \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + a_2 \cdot \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} + \cdots + a_n \cdot \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} = a_1 \mathbf{e}_1 + a_2 \mathbf{e}_3 + \cdots + a_n \mathbf{e}_n,$$

for all $a_1, a_2, \ldots, a_n \in \mathbb{R}$. 

Lecture 17: Linear Independence, Bases and Dimension
Properties of the standard basis

(i) No vector in the standard basis can be written as a linear combination of the remaining standard basis vectors.

Equivalently:

\[ e_i \not\in \text{span}\{e_1, \ldots, e_{i-1}, e_{i+1}, \ldots, e_n\}, \]

for all \( 1 \leq i \leq n \).

(ii) If we have a linear combination

\[ \lambda_1 e_1 + \lambda_2 e_2 + \cdots + \lambda_n e_n = 0, \]

then \( \lambda_1 = \lambda_2 = \cdots = \lambda_n = 0 \).

(iii) Any vector in \( \mathbb{R}^n = \text{span}\{e_1, \ldots, e_n\} \) can be written uniquely as a linear combination of \( e_1, \ldots, e_n \).

In other words: If

\[ \alpha_1 e_1 + \cdots + \alpha_n e_n = \beta_1 e_1 + \cdots + \beta_n e_n, \]

then \( \alpha_1 = \beta_1, \ldots, \alpha_n = \beta_n \).
Theorem

**Very Important Theorem.** Let $v_1, \ldots, v_r$ be vectors in $\mathbb{R}^n$ and let $A$ denote the $n \times r$ matrix $A = [v_1 \ v_2 \ \cdots \ v_r]$. The following equivalent conditions are equivalent:

(i) No vector in the list can be written as a linear combination of the remaining vectors in the list.

(ii) If we have a linear combination

$$\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_r v_r = 0,$$

then $\lambda_1 = \lambda_2 = \cdots = \lambda_r = 0$.

(iii) Any vector in $\text{span}\{v_1, \ldots, v_r\}$ can be written **uniquely** as a linear combination of $v_1, \ldots, v_r$.

(iv) The system of equations $A \cdot X = 0$ has only the 0 solution.

Note that condition (iv) is equivalent to the conditions (i)-(iii) since

$$A \cdot \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_r \end{bmatrix} = 0$$

gives a solution to (iv) if and only if $\lambda_1 v_1 + \cdots \lambda_r v_r = 0$. 

Lecture 17: Linear Independence, Bases and Dimension
Definition

Important Definition:

Vectors $v_1, \ldots, v_r$ in $\mathbb{R}^n$ are said to be \textit{linearly independent} if any of the four equivalent conditions in the theorem above hold.
Examples

Example 1: $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ are linearly independent.

Example 2: The vectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ are linearly independent.

We must show that for $A = [\mathbf{v}_1 \ \mathbf{v}_2 \ \mathbf{v}_3]$, the system $A \cdot \mathbf{X} = \mathbf{0}$ has only the zero solution.

Solution:

\[
\begin{bmatrix}
1 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 \\
0 & 1 & 1 & 0
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 1 & 0 & 0 \\
0 & -1 & 1 & 0 \\
0 & 1 & 1 & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 0 & 1 & 0 \\
0 & -1 & 1 & 0 \\
0 & 0 & 2 & 0
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix}
\]

which shows that the system $A \cdot \mathbf{X} = \mathbf{0}$ has a unique solution. Therefore, the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent.
Show that the vectors \( v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \ v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \ v_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \) are linearly independent.

Solution: For \( A = [v_1 \ v_2 \ v_3] \), we check \( A \cdot X = 0 \) has only \( 0 \) as a solution.

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 \\
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & -1 & 0
\end{bmatrix}
\]

which shows that the systems \( A \cdot X = 0 \) has only \( 0 \) as a solution.

Thus \( v_1, \ldots, v_r \) are linearly independent.
Definition

Vectors \( v_1, \ldots, v_r \in \mathbb{R}^n \) are **linearly dependent** if they are not linearly independent.

In particular: Vectors \( v_1, \ldots, v_r \) are linearly dependent if one of the following equivalent conditions hold:

(i) Some \( v_i \) is in the span of \( v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_r \).

(ii) There exists a non-trivial dependence relation:

\[
\lambda_1 v_1 + \cdots + \lambda_r v_r = 0
\]

with NOT all \( \lambda_i = 0 \).

(iii) For \( A = [v_1 \cdots v_r] \), there is a non-zero solution to \( A \cdot X = 0 \).

Observation: Suppose \( \lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 = 0 \), with say, \( \lambda_2 \neq 0 \). Then

\[
-\lambda_2 v_2 = \lambda_1 v_1 + \lambda_3 v_3,
\]

so \( v_2 = -\frac{\lambda_1}{\lambda_2} v_1 + -\frac{\lambda_3}{\lambda_1} v_3 \). This shows how a dependence relation among the vectors \( v_i \) leads to expressing one of the vectors in terms of the others.
The vectors \( \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \), \( \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \), \( \mathbf{v}_3 = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} \) are linearly dependent.

Find a dependence relation among them and use it to express one of the vectors as a linear combination of the remaining vectors.

Solution: If the vectors are not linearly independent, then there is a non-zero solution to the system \( A\mathbf{x} = \mathbf{0} \), where \( A = [\mathbf{v}_1 \ \mathbf{v}_2 \ \mathbf{v}_3] \).

\[
\begin{bmatrix}
1 & 1 & 3 & 0 \\
1 & 2 & 4 & 0 \\
2 & 1 & 5 & 0
\end{bmatrix}
\xrightarrow{-R_1+R_2}
\begin{bmatrix}
1 & 1 & 3 & 0 \\
0 & 1 & 1 & 0 \\
2 & 1 & 5 & 0
\end{bmatrix}
\xrightarrow{-2\cdot R_1+R_3}
\begin{bmatrix}
1 & 1 & 3 & 0 \\
0 & -1 & -1 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\xrightarrow{R_2+R_3}
\begin{bmatrix}
1 & 0 & 2 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}.
\]

We can write the solution to the homogeneous system as
\[
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix}
= \begin{bmatrix} -2t \\
-t \\
t
\end{bmatrix}
= t \cdot \begin{bmatrix} -2 \\
-1 \\
1
\end{bmatrix}.
\]
This shows that $-2 \cdot v_1 + (-1) \cdot v_2 + v_3 = 0$. Thus, $v_3 = 2 \cdot v_1 + v_2$.

CHECK: $2 \cdot v_1 + v_2 = 2 \cdot \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$, as required.

Since any multiple of $\begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$ is also a solution, any such multiple gives a dependence relation on $v_1, v_2, v_3$. For example, taking $t = -3$, we get $\begin{bmatrix} 6 \\ 3 \\ -3 \end{bmatrix}$ is a solution, so that $6 \cdot v_1 + 3 \cdot v_2 - 3 \cdot v_3 = 0$.

Thus there are infinitely many dependence relations among $v_1, v_2, v_3$. But in this case, just one way to write $v_3$ as a linear combination of $v_1$ and $v_2$. 
Determine whether or not the vectors \( v_1 = \begin{bmatrix} 4 \\ -1 \\ -3 \end{bmatrix} \), \( v_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \), \( v_3 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \) are linearly independent or linearly dependent. If they are linearly dependent, express one of the vectors as a linear combination of the remaining vectors.

Solution: We take \( A = [v_1 \ v_2 \ v_3] \). If the only solution to \( A \cdot X = 0 \) is \( 0 \), the vectors are linearly independent.

If \( A \cdot X = 0 \) has a non-zero solution, the vectors are linearly dependent, and any non-zero solution gives a dependence relation.
Thus, the solution to the homogeneous system is \[ \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -t \\ 4t \\ t \end{bmatrix}, \] which shows that \( v_1, v_2, v_3 \) are linearly dependent.

Taking \( t = 1 \) yields the dependence relation \(-v_1 + 4v_2 + v_3 = 0\).

Thus, \( v_1 = 4v_2 + v_3 \).
CHECK:

\[ 4v_2 + v_3 = 4 \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ -4 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ -3 \end{bmatrix}, \]

as required.
Important Definition: Let $U \subseteq \mathbb{R}^n$ be a subspace of $\mathbb{R}^n$. Vectors $v_1, \ldots, v_r \in U$ are a basis for $U$ if:

(i) $U = \text{span}\{v_1, \ldots, v_r\}$.

(ii) The vectors $v_1, \ldots, v_r$ are linearly independent.

In particular: A basis for $\mathbb{R}^n$ is a collection of linearly independent vectors that span $\mathbb{R}^n$.

Moreover: If $v_1, \ldots, v_n$ is a basis for $\mathbb{R}^n$, then: Every vector in $\mathbb{R}^n$ can be written *uniquely* as a linear combination of $v_1, \ldots, v_n$.

Examples: (i) The standard basis $e_1, e_2, \ldots, e_n$ is a basis for $\mathbb{R}^n$.

(ii) The basic solutions to a homogeneous system of linear equations form a basis for the solution space of that system.

(iii) If $\lambda$ is an eigenvalue for the matrix $A$, then the basic $\lambda$-eigenvectors form a basis for $E_\lambda$, the eigenspace of $\lambda$. 
Example

Find a basis for the subspace of \( \mathbb{R}^3 \) that is the solution space to the homogeneous equation:

\[
2x - 4y + 10z = 0.
\]

Note that this solution space is a plane through the origin in \( \mathbb{R}^3 \).

Solution: \( x = 2y - 5z \). Thus, in vector form, the solutions are given by

\[
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix} = \begin{bmatrix}
2s - 5t \\
s \\
t
\end{bmatrix} = s \cdot \begin{bmatrix}
2 \\
1 \\
0
\end{bmatrix} + t \cdot \begin{bmatrix}
-5 \\
0 \\
1
\end{bmatrix}.
\]

Thus \( v_1 = \begin{bmatrix}
2 \\
1 \\
0
\end{bmatrix} \) and \( v_2 = \begin{bmatrix}
-5 \\
0 \\
1
\end{bmatrix} \) are basic solutions.
Example continued

To see that the basic solutions are in fact a basis for the solution space, note that the vector equation shows that the basic solutions span the solution space.

In addition, if \( \alpha_1 \cdot v_1 + \alpha_2 \cdot v_2 = 0 \), then:

\[
\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix} = \alpha \cdot \begin{bmatrix}
2 \\
1 \\
0
\end{bmatrix} + \alpha_2 \cdot \begin{bmatrix}
-5 \\
0 \\
1
\end{bmatrix} = \begin{bmatrix}
2\alpha_1 - 5\alpha_2 \\
\alpha_1 \\
\alpha_2
\end{bmatrix},
\]

which gives: \( \alpha_1 = \alpha_2 = 0 \).

Thus \( v_1, v_2 \) are linearly independent and therefore form a basis for the solutions space, or equivalently, a basis for the given plane through the origin.
Class Example

The matrix $A = \begin{bmatrix} 2 & 2 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ has 2 as an eigenvalue of multiplicity two.

Find a basis for the eigenspace $E_2$.

Solution: We find the solutions to the homogeneous system having

$$2I_3 - A = \begin{bmatrix} 0 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

as its coefficient matrix.

This matrix clearly reduces to

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

For such a homogeneous system, $y = 0$, while $x$ and $z$ are free variables. Thus the solutions are:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} s \\ 0 \\ t \end{bmatrix} = s \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$
Thus, the basic solutions to the homogeneous system with coefficient matrix $2I_3 - A$ are:

\[
\begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix},
\]

which in turn form a basis for the eigenspace $E_2$. 

Class Example continued
Very Important Fact. Suppose the subspace $U$ of $\mathbb{R}^n$ is spanned by the vectors $v_1, \ldots, v_r$. Then there exists a subset of $v_1, \ldots, v_r$ forming a basis of $U$.

Why: Suppose $U = \text{span}\{v_1, v_2, v_3, v_4\}$. If $v_1, \ldots, v_4$ are linearly independent, they form a basis for $U$.

Otherwise of one the vectors is in the span of the remaining ones: say, $v_2 = av_1 + bv_3 + cv_4$.

Suppose $u \in U$. We can write

$$u = pv_1 + qv_2 + rv_3 + sv_4 = pv_1 + q(av_1 + bv_3 + cv_4) + rv_3 + sv_4$$

$$= (p + aq)v_1 + (qb + r)v_3 + (qc + s)v_4.$$ 

Thus, $u \in \text{span}\{v_1, v_3, v_4\}$. Thus: $U = \text{span}\{v_1, v_3, v_4\}$.

If $v_1, v_3, v_4$ are linearly independent, they form a basis for $U$. Otherwise, we may eliminate another vector and continue the process until we have a linearly independent spanning set for $U$, that is, a basis for $U$. 

Lecture 17: Linear Independence, Bases and Dimension
Fundamental Theorem

Let $v_1, \ldots, v_r$ be vectors in $\mathbb{R}^n$ that span the subspace $U$ and suppose $w_1, \ldots, w_t \in U$ are linearly independent. Then:

(i) $t \leq r$.

(ii) Any two bases for $U$ have the same number of elements.

The **dimension** of $U$ is the number of elements in any basis of $U$.

**Corollary.** The dimension of $\mathbb{R}^n$ equals $n$.

WHY: $e_1, e_2, \ldots, e_n$ forms a basis for $\mathbb{R}^n$. 